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Brane dualities in non-relativistic limit

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Abstract: We analyze brane dualities in the non-relativistic limit of the worldvolume actions. In particular we have analyzed how the non-relativistic M2-brane is related via these dualities to non-relativistic D2-brane, non-relativistic IIA fundamental string and also, by using T-duality, to non-relativistic D1-string. These actions coincide with ones obtained from relativistic actions by taking non-relativistic limit, showing that the non-relativistic limit and the dualities commute in these cases.

Keywords: D-branes, AdS-CFT Correspondence, String Duality.

Contents

1. Introduction

Non-relativistic limit of string theories appear to be solvable sectors [1, 2]. It can provide us a deeper understanding of string theories and can be helpful to answer some of open questions as for example AdS/CFT correspondence. In [3 − 6] the non-relativistic limits of strings and branes have been analyzed using the worldvolume actions. They have the Galilean supersymmetries and the non-relativistic spectra. A notable property is that the kappa symmetry can be maintained in the non-relativistic limit. By fixing the kappa gauge choice suitably these systems become free in the static gauge .

In this paper we examine some of dualities between branes and strings in the non-relativistic limit using their worldvolume actions. It is interesting to see how these dualities remain in the non-relativistic limit [1]. Here we study the dualities at the worldvolume actions by considering the non-relativistic M2-brane as a starting point to obtain non-relativistic type IIA superstring and non-relativistic D2-brane, using analogous procedures to the relativistic case [7, 8]. We also examine the T-duality between D2 and D1 branes [9, 10] and find that the T-duality transformations are compatible with the non-relativistic limit. In the cases we examine in this paper we see the dualities remain in the non-relativistic sector of the string theories, (figure 1).

The paper is organized as follows. In section 2, we obtain the non-relativistic p-brane worldvolume action of Polyakov type. In section 3 we consider a reduction from non-relativistic M2-brane to non-relativistic D2-brane by dualizing the 11-th coordinate in the

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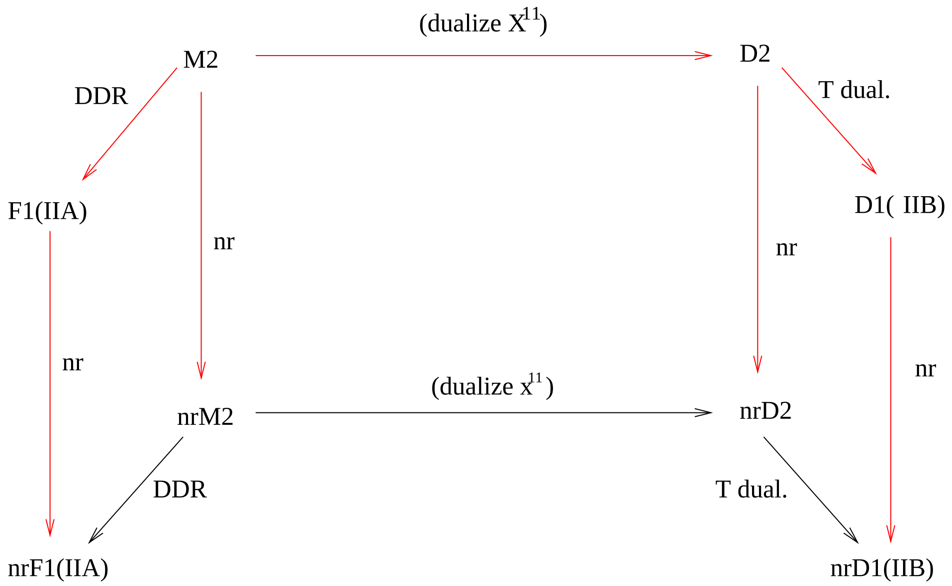

Figure 1: This diagram shows the different ways in which the non-relativistic type IIA string, non-relativistic D2-brane and non-relativistic D1-string are obtained. Due to the commutativity of the non-relativistic limit and the dualities, we can arrive at the same actions by taking in different paths.

same way as in the relativistic case. In section 4 we discuss the double dimensional reduction of the non-relativistic M2 brane to obtain the non-relativistic type IIA superstring. In section 5 we analyze the T-duality transformation of the non-relativistic D2-brane to non-relativistic D1-string. Summary is in the last section and appendices are attached for notations and some detail.

2. NR limit of p-branes

The non-relativistic limit of the p-branes have been discussed using their worldvolume actions for the bosonic branes [3] and the supersymmetric ones [4]. It is established how to obtain worldvolume actions of non-relativistic branes from the relativistic ones of Nambu-Goto form. Here we make a short summary but starting from Polyakov form action and we find the non-relativistic p-brane action of Polyakov form.

We start by considering the supersymmetric p-brane action in a flat D dimensional background,

$$S^p = T_p \int d^{p+1}\xi\, \mathcal{L}, \qquad \mathcal{L} = \mathcal{L}^P + \mathcal{L}^{WZ} + \mathcal{L}^B, \tag{2.1}$$

where T_p is the p-brane tension. The kinetic Lagrangian has the Polyakov form using the worldvolume metric $\gamma_{IJ}, (I = 0, 1, \ldots, p)$,

$$\mathcal{L}^P = -\frac{\sqrt{-\gamma}}{2}\left(\gamma^{IJ}G_{IJ} - (p-1)\right). \tag{2.2}$$

The WZ Lagrangian is the pullback of $p+1$ form b determined from a closed and super-invariant $p+2$ form h. For minimal spinor θ in the D dimensions,

$$\mathcal{L}^{WZ}{=}b^*, \qquad h = db = -\frac{i}{p!}\, d\bar{\theta}\not{\Pi}^p\, d\theta, \tag{2.3}$$

where the super invariant one forms $\Pi^M, (M = 0, \ldots, D-1)$, and the induced metric G_{IJ} are

$$\Pi^M{=}dX^M + i\bar{\theta}\Gamma^M d\theta \equiv d\xi^I \Pi_I{}^M, \qquad G_{IJ}{=}\Pi_I{}^M\Pi_J{}^N\eta_{MN}. \tag{2.4}$$

To show its closure we need the gamma matrix identity (A.3) valid in the D dimensions where p-branes exist [11] (See appendix A for detailed notations). For the M2-brane we will discuss in this paper the explicit form of b is

$$b = -\frac{1}{2}(i\bar{\theta}\Gamma_{ML}d\theta)\left(\Pi^M\Pi^L - (i\bar{\theta}\Gamma^M d\theta)\Pi^L + \frac{1}{3}(i\bar{\theta}\Gamma^M d\theta)(i\bar{\theta}\Gamma^L d\theta)\right). \tag{2.5}$$

$\mathcal{L}^B$ is the B-field Lagrangian and is the pullback of a closed B-field. It is introduced without modifying the supergravity equations of motion.

In ref. [4] we have shown the non-relativistic p-brane worldvolume Lagrangian is obtained from the relativistic one by rescaling of the supercoordinates and the tension as

$$X^\mu = \omega x^\mu, \qquad X^A = x^A, \qquad \theta = \sqrt{\omega}\theta_- + \frac{1}{\sqrt{\omega}}\theta_+, \qquad T_p = \omega^{1-p}\tilde{T}_p. \tag{2.6}$$

and then taking $\omega \to \infty$. In the non-relativistic limit, the longitudinal coordinates $X^\mu, (\mu = 0, 1, \ldots, p)$ of the string are rescaled while the transverse ones $X^A, (A = p+1, \ldots, D-1)$ are left untouched, which implies that the transverse fluctuations are small. The scaling behavior of the fermionic coordinates in (2.6) is characterized by their transformation properties under the matrix Γ_*, which splits the fermionic coordinates into two eigenspaces $\theta_\pm$ of Γ_* with eigenvalues ± 1,

$$\theta_\pm \equiv \mathbb{P}_\pm\theta, \qquad \mathbb{P}_\pm = \frac{1}{2}(1 \pm \Gamma_*), \qquad \Gamma_* = \Gamma_0 \ldots \Gamma_{\rm p}. \tag{2.7}$$

In the Polyakov form we should also rescale the worldvolume metric γ_{IJ} as

$$\gamma_{IJ} = \omega^2\tilde{\gamma}_{IJ}. \tag{2.8}$$

In the $\omega \to \infty$ limit the Lagrangian has divergent terms of order ω^2. They can be cancelled by adjusting the divergent contribution of the B-field Lagrangian $\mathcal{L}^B$ in (2.1) in the non-relativistic limit,

$$\mathcal{L}^B \equiv T_p \det(\partial_I X^\mu) = \omega^2\tilde{T}_p \det(\partial_I x^\mu) \equiv \omega^2\, \tilde{T}_p\, \mathcal{L}^B_{\rm div}. \tag{2.9}$$

Its contribution to the energy is precisely compensated with the contribution coming from the tension of the brane. Under these choices the action in the $\omega \to \infty$ is finite and shown to have a non-relativistic supersymmetry and kappa invariance.

Making these rescalings in the Lagrangian (2.1) we get

$$S^p = \tilde{T}_p \int d^{p+1}\xi\ \left(\omega^2(\mathcal{L}^P_{\rm div} + \mathcal{L}^{WZ}_{\rm div} + \mathcal{L}^B) + (\mathcal{L}^P_{\rm fin} + \mathcal{L}^{WZ}_{\rm fin})\right), \tag{2.10}$$

where

$$\mathcal{L}^P_{\rm div} = -\frac{\sqrt{-\tilde{\gamma}}}{2}\left(\tilde{\gamma}^{IJ} g_{IJ} - (p-1)\right), \tag{2.11}$$

$$\mathcal{L}^P_{\rm fin} = -\frac{\sqrt{-\tilde{\gamma}}}{2}\,\tilde{\gamma}^{IJ} G^{nr}_{IJ} \tag{2.12}$$

and[1]

$$\begin{aligned} g_{IJ} &= e_I{}^\mu e_J{}^\nu \eta_{\mu\nu}, \qquad e^\mu = dx^\mu + i\bar{\theta}_-\Gamma^\mu d\theta_-, \\ G^{nr}_{IJ} &= u_I{}^A u_J{}^B \delta_{AB} + 2i\bar{\theta}_+\Gamma_\mu e_{(I}{}^\mu \partial_{J)}\theta_+, \quad u^A = dx^A + (i\bar{\theta}_-\Gamma^A d\theta_+ + i\bar{\theta}_+\Gamma^A d\theta_-). \end{aligned} \tag{2.13}$$

Here and hereafter inverse power terms of ω, which do not contribute in the $\omega \to \infty$ limit eventually, are omitted. The sum of divergent terms of the WZ and B field Lagrangians is shown to be

$$\mathcal{L}^{WZ}_{\rm div} + \mathcal{L}^B = \det(e_I{}^\mu) = \sqrt{-g}. \tag{2.14}$$

By adding $\mathcal{L}^P_{\rm div}$ the ω^2 terms of the total Lagrangian is

$$\omega^2 \mathcal{L}_{\rm div} = \omega^2\left[-\frac{\sqrt{-\tilde{\gamma}}}{2}\left(\tilde{\gamma}^{IJ} g_{IJ} - (p-1)\right) + \sqrt{-g}\,\right]. \tag{2.15}$$

It is straightforward to show that it has an expansion of $(\tilde{\gamma}_{IJ} - g_{IJ})$ starting from their bi-linear terms,

$$\omega^2 \mathcal{L}_{\rm div} = -\frac{\omega^2}{2} A^{IJ,KL}(\tilde{\gamma}_{IJ} - g_{IJ})(\tilde{\gamma}_{KL} - g_{KL}), \tag{2.16}$$

where[2]

$$A^{IJ,KL}|_{\tilde{\gamma}_{IJ}=g_{IJ}} = \tilde{T}_p \frac{\sqrt{-g}}{4}(g^{IK}g^{JL} + g^{IL}g^{JK} - g^{IJ}g^{KL}). \tag{2.17}$$

We can now rewrite this superficially divergent term in a way that the $\omega \to \infty$ limit can be taken smoothly. The idea is to introduce Lagrange multipliers to rewrite the action. For the case $p \neq 1$ it is

$$\omega^2 \mathcal{L}_{\rm div} = \lambda^{IJ}\,(\tilde{\gamma}_{IJ} - g_{IJ}) + \frac{1}{2\omega^2} A^{-1}_{IJ,KL}\lambda^{IJ}\lambda^{KL}, \tag{2.18}$$

where

$$\begin{aligned} A^{-1}_{IJ,RS} A^{RS,KL} &= \delta_I{}^{(K}\delta_J{}^{L)}, \\ A^{-1}_{IJ,KL}|_{\gamma_{IJ}=g_{IJ}} &= \tilde{T}_p \frac{2}{\sqrt{-g}}\left(g_{IK}g_{JL} + g_{IL}g_{JK} - \frac{2}{p-1} g_{IJ}g_{KL}\right). \end{aligned} \tag{2.19}$$

[1] The symmetrization conventions here are $A_{(i}B_{j)} = \frac{1}{2}(A_iB_j + A_jB_i)$, $A_{[i}B_{j]} = \frac{1}{2}(A_iB_j - A_jB_i)$.

[2] We only need to consider A and A^{-1} in the vicinity of $\tilde{\gamma}_{IJ} = g_{IJ}$. Corrections are proportional to $(\tilde{\gamma} - g)^2$ and turn to vanish in the $\omega \to \infty$ limit.

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(2.18) reproduces (2.16) by integrating out the λ^{IJ} variables. We can take the strict non-relativistic limit $\omega \to \infty$ in (2.18) and be left with a finite contribution:

$$\mathcal{L}^* = \lambda^{IJ}(\tilde{\gamma}_{IJ} - g_{IJ}). \tag{2.20}$$

For the $p = 1$ case $A^{IJ,KL}$ is singular and a device is required. Following to [5] we write

$$\omega^2 \mathcal{L}_{\rm div} = -\omega^2 \frac{\tilde{T}_p}{2}\sqrt{-\tilde{\gamma}}\tilde{\gamma}^{00}\tilde{\gamma}_{\hat{\mu}\hat{\nu}}\, f^{\hat{\mu}} f^{\hat{\nu}}, \tag{2.21}$$

where $\hat{\mu} = 0, 1$ and

$$f^{\hat{\mu}} \equiv \left[e_0{}^{\hat{\mu}} - \frac{\sqrt{-\tilde{\gamma}}}{\tilde{\gamma}_{11}}\varepsilon^{\hat{\mu}\hat{\rho}}\eta_{\hat{\rho}\hat{\sigma}}e_1{}^{\hat{\sigma}} - \frac{\tilde{\gamma}_{01}}{\tilde{\gamma}_{11}}e_1{}^{\hat{\mu}} \right] \tag{2.22}$$

and rewrite the superficially divergent term as

$$\mathcal{L}_{\rm div} = \lambda_{\hat{\mu}} f^{\hat{\mu}} + \frac{1}{2\omega^2 \tilde{T}_p \sqrt{-\tilde{\gamma}}\tilde{\gamma}^{00}}\lambda_{\hat{\mu}}\lambda_{\hat{\nu}}\eta^{\hat{\mu}\hat{\nu}}, \tag{2.23}$$

which reproduces (2.21) by integrating out the $\lambda_{\hat{\mu}}$ variables. It becomes in the non-relativistic limit $\omega \to \infty$

$$\mathcal{L}^* = \lambda_{\hat{\mu}} f^{\hat{\mu}}. \tag{2.24}$$

Now that we have properly defined the superficial divergence, we can finally write the complete finite action for non-relativistic p-branes as

$$S^p_{nr} = \tilde{T}_p \int d^{p+1}\xi \left(\mathcal{L}^P_{\rm fin} + \mathcal{L}^{WZ}_{\rm fin} + \mathcal{L}^* \right) = \tilde{T}_p \int d^{p+1}\xi \, \left(-\frac{\sqrt{-\tilde{\gamma}}}{2}\tilde{\gamma}^{IJ}G^{nr}_{IJ} + \mathcal{L}^{WZ}_{\rm fin} + \mathcal{L}^* \right). \tag{2.25}$$

In contrast to the relativistic case, the non-relativistic Polyakov Lagrangian (2.25) contains extra multiplier variables λ's. As was discussed in [1] for the string case, they are indispensable to realize the non-relativistic symmetry and the spectrum upon quantization.

The Nambu-Goto form of the non-relativistic p-brane action can be obtained by eliminating the worldvolume metric $\tilde{\gamma}_{IJ}$ using the equations of motion with respect to $\lambda^{IJ}, (p \neq 1)$ or $\lambda_{\hat{\mu}}, (p = 1)$ cases,

$$\tilde{\gamma}_{IJ} = g_{IJ}, \ (p \neq 1), \quad \text{and} \quad \tilde{\gamma}_{IJ} \propto g_{IJ}, \ (p = 1). \tag{2.26}$$

The action (2.25) becomes

$$S^p_{nr} = \tilde{T}_p \int d^{p+1}\xi \ \left(\mathcal{L}^{NG}_{\rm fin} + \mathcal{L}^{WZ}_{\rm fin} \right) = \tilde{T}_p \int d^{p+1}\xi \, \left(-\frac{\sqrt{-g}}{2}\, g^{IJ}G^{nr}_{IJ} + \mathcal{L}^{WZ}_{\rm fin} \right). \tag{2.27}$$

The finite part of the WZ Lagrangian $\mathcal{L}^{WZ}_{\rm fin}$ is the pullback of the ω^{p-1}-th order term b_{nr} of $p+1$ form b in the non-relativistic expansion. It is obtained by integrating ω^{p-1}-th order term of h in (2.3). The explicit form of b_{nr} for the non-relativistic M2-brane ($p = 2$) is given as [4],

$$b^{M2}_{nr} = -\frac{1}{2}\Bigg\{ K^+_{\mu\nu}\left[e^\mu e^\nu - K^\mu_- e^\nu + \frac{1}{3}K^\mu_- K^\nu_- \right] + K^-_{\mu\nu}K^\mu_+ \left[e^\nu - \frac{1}{3}K^\nu_- \right]$$

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$$+ L_{\mu B}\left[2e^\mu u^B - e^\mu L^B - K_-^\mu u^B + \frac{2}{3}K_-^\mu L^B\right]$$
$$+ K^-_{AB}\left[u^A u^B - L^A u^B + \frac{1}{3}L^A L^B\right]\Bigg\}, \tag{2.28}$$

where

$$K^\pm_{\mu\nu} = i\bar\theta_\pm\Gamma_{\mu\nu}d\theta_\pm, \qquad K^\mu_\pm = i\bar\theta_\pm\Gamma^\mu d\theta_\pm, \qquad K^-_{AB} = i\bar\theta_-\Gamma_{AB}d\theta_-,$$
$$L_{\mu A} = i\bar\theta_+\Gamma_\mu\Gamma_A d\theta_- + i\bar\theta_-\Gamma_\mu\Gamma_A d\theta_+, \qquad L^A = i\bar\theta_+\Gamma^A d\theta_- + i\bar\theta_-\Gamma^A d\theta_+. \tag{2.29}$$

3. Non-relativistic M2 to non-relativistic D2 brane

In [8] the D2-brane action is obtained by dualizing 11-th coordinate X^{11} of the M2-brane to the BI U(1) potential. In this section we show that the non-relativistic D2-brane action is obtained from non-relativistic M2-brane by dualizing 11-th coordinate of the non-relativistic M2-brane in the same manner as in the relativistic case. As the non-relativistic D2-brane action is obtained by taking the non-relativistic limit of the relativistic one [6], the non-relativistic limit and the dualization are commutative.

We first summarize briefly the result of [8]. In order to obtain D2-brane from M2-brane, the U(1) gauge field degrees of freedom must be introduced. There is an extra coordinate X^{11} in the M2 brane which appears in the action in a form $\varphi = dX^{11}$. By introducing one form Lagrange multiplier A and adding a term $T_2\varphi dA$ on the Lagrangian the dX^{11} can be replaced by φ. The one form φ can be regarded as an independent variable. The variation of the Lagrangian with respect to the A gives an equation of motion $d\varphi = 0$ whose solution is $\varphi = d\chi$. The M2-brane system is reproduced by identifying the χ with X^{11}. On the other hand if we eliminate the φ by its equation of motion we get a Lagrangian depending on the A. Since the A enters in the Lagrangian only through $F = dA$ the A becomes a U(1) gauge field. The resulting Lagrangian is shown to describe the D2-brane system.

More explicitly we start with the M2-brane action of Polyakov form (2.1). Replacing $\partial_I X^{11}$ by φ_I,

$$S^{M2} = -T_2\int d^3\xi\frac{\sqrt{-\gamma}}{2}\left\{\gamma^{IJ}\Pi^m_I\Pi^n_J\eta_{mn} + \gamma^{IJ}\left(\varphi_I + i\bar\theta\Gamma_{11}\partial_I\theta\right)\left(\varphi_J + i\bar\theta\Gamma_{11}\partial_J\theta\right) - 1\right\}$$
$$+T_2\int\left\{-C^{(3)} - b^{(2)}\varphi + \varphi\, dA\right\}, \tag{3.1}$$

where the last term is the Lagrangian multiplier term and $m = 0,\ldots,9$. The WZ three form b of the M2-brane (2.5) is split into a term $C^{(3)}$ independent of dX^{11} and a term linear in dX^{11} as

$$b \equiv -C^{(3)} - b^{(2)}dX^{11}, \tag{3.2}$$

where

$$C^{(3)} = \frac{1}{2}(i\bar\theta\Gamma_{mn}d\theta)\left(\Pi^m\Pi^n - (i\bar\theta\Gamma^m d\theta)\Pi^n + \frac{1}{3}(i\bar\theta\Gamma^m d\theta)(i\bar\theta\Gamma^n d\theta)\right)$$
$$+\frac{1}{2}(i\bar\theta\Gamma_m\Gamma_{11}d\theta)(\Pi^m - \frac{1}{3}i\bar\theta\Gamma^m d\theta)(i\bar\theta\Gamma^{11}d\theta), \tag{3.3}$$

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$$b^{(2)} = (i\bar{\theta}\Gamma_m\Gamma_{11}d\theta)(\Pi^m - \frac{1}{2}i\bar{\theta}\Gamma^m d\theta). \tag{3.4}$$

By integrating out the φ_I, the action becomes that of the D2-brane,

$$\begin{aligned} S^{D2} &= -T_2\int d^3\xi\frac{\sqrt{-\gamma}}{2}\left\{\gamma^{IJ}\Pi_I^m\Pi_J^n\eta_{mn} - 1 + \frac{1}{2}\gamma^{IK}\gamma^{JL}\mathcal{F}_{IJ}\mathcal{F}_{KL}\right\} \\ &\quad + T_2\int\left\{-C^{(3)} - C^{(1)}\,\mathcal{F}\right\}, \end{aligned} \tag{3.5}$$

where

$$\mathcal{F} = dA - b^{(2)}, \qquad \text{and} \qquad C^{(1)} = i\bar{\theta}\Gamma_{11}d\theta. \tag{3.6}$$

$\mathcal{F}$ is the superinvariant U(1) field strength modified by the NS two form $b^{(2)}$. $C^{(3)}$ and $C^{(1)}$ are the RR potential forms in the flat supergravity background. The second line of (3.5) is giving the WZ Lagrangian of the D2-brane satisfying

$$d\left(C^{(3)} + C^{(1)}\mathcal{F}\right) = id\bar{\theta}\left(\frac{\Gamma_{mn}\Pi^m\Pi^n}{2!} + \Gamma_{11}\mathcal{F}\right)d\theta, \tag{3.7}$$

which is the condition required for the Dp-brane actions [12−15].

Eq. (3.5) is the D2-brane action of the Polyakov type. It has been shown [8] perturbatively in $\mathcal{F}$ that it gives the Dirac-Born-Infeld action by an integration of the worldvolume metric γ_{IJ},

$$S^{BI} = -T_2\int d^3\xi\sqrt{-\det(G_{IJ}^{D2} + \mathcal{F}_{IJ})} - T_2\int\left(C^{(3)} + C^{(1)}\mathcal{F}\right), \tag{3.8}$$

where G_{IJ}^{D2} is the induced metric of D2-brane

$$G_{IJ}^{D2} = \Pi_I^m\Pi_J^n\eta_{mn}. \tag{3.9}$$

In the appendix B we show it to all order of $\mathcal{F}$.

The non-relativistic D2-brane worldvolume action is obtained from that of the non-relativistic M2-brane. We begin with the non-relativistic action of M2-brane (2.27). In contrast to the relativistic case we can use the NG form in which the worldvolume metric has been eliminated,

$$S_{nr}^{M2} = -\tilde{T}_2\int d^3\xi\frac{\sqrt{-g}}{2}\left\{g^{IJ}u_I^Au_J^B\eta_{AB} + 2i\bar{\theta}_+\Gamma^\mu e_\mu{}^I\partial_I\theta_+\right\} + \tilde{T}_2\int b_{nr}^{M2}. \tag{3.10}$$

In the WZ Lagrangian the b_{nr}^{M2} is given in (2.28) as

$$b_{nr}^{M2} = -C_{nr}^{(3)} - b_{nr}^{(2)}dx^{11}, \tag{3.11}$$

where $C_{nr}^{(3)}$ is the RR three form potential and $b_{nr}^{(2)}$ is the NS two form in the non-relativistic limit

$$C_{nr}^{(3)} = \frac{1}{2}\left\{K_{\mu\nu}^+\left[e^\mu e^\nu - K_-^\mu e^\nu + \frac{1}{3}K_-^\mu K_-^\nu\right] + K_{\mu\nu}^- K_+^\mu\left[e^\nu - \frac{1}{3}K_-^\nu\right]\right.$$

$$+L_{\mu b}\left[2e^\mu u^b - e^\mu L^b - K_-^\mu u^b + \frac{2}{3}K_-^\mu L^b\right] + K_{ab}^-\left[u^a u^b - L^a u^b + \frac{1}{3}L^a L^b\right]$$

$$+L_{\mu 11}\left[e^\mu - \frac{1}{3}K_-^\mu\right]L^{11} + K_{b11}^-\left[u^b - \frac{1}{3}L^b\right]L^{11}\Bigg\},$$

$$b_{nr}^{(2)} = L_{\mu 11}(e^\mu - \frac{1}{2}K_-^\mu) + K_{a11}^-(u^a - \frac{1}{2}L^a), \tag{3.12}$$

where the fermionic bi-linears K's and L's are given in (2.29). We separate terms depending on dx^{11} and replace dx^{11} with φ by adding a Lagrange multiplier term $\varphi d\tilde{A}$ in the action

$$S_{nr}^{M2} = -\tilde{T}_2\int d^3\xi\frac{\sqrt{-g}}{2}\left\{g^{IJ}{u_I}^a{u_J}^b\eta_{ab} + g^{IJ}(\varphi + C_{nr}^{(1)})_I(\varphi + C_{nr}^{(1)})_J + 2i\bar{\theta}_+\Gamma^\mu {e_\mu}^I\partial_I\theta_+\right\}$$
$$+\tilde{T}_2\int\left(-C_{nr}^{(3)} - b_{nr}^{(2)}\varphi + \varphi\, d\tilde{A}\right), \tag{3.13}$$

where $a, b = 3, \ldots, 9$. $C_{nr}^{(1)}$ is RR one form potential equal to L^{11} in (2.29),

$$C_{nr}^{(1)} = L^{11} = i\bar{\theta}_+\Gamma^{11}d\theta_- + i\bar{\theta}_-\Gamma^{11}d\theta_+ \tag{3.14}$$

and $(\varphi + C_{nr}^{(1)})_I$ is the I-th component of the one form $(\varphi + C_{nr}^{(1)})$. By integrating out the φ_I we arrive at the non-relativistic D2-brane action,

$$S_{nr}^{D2} = -\tilde{T}_2\int d^3\xi\frac{\sqrt{-g}}{2}\left\{g^{IJ}G_{nr;IJ}^{D2} + \frac{1}{2}g^{IK}g^{JL}\mathcal{F}_{nr;IJ}\mathcal{F}_{nr;KL}\right\}$$
$$+\tilde{T}_2\int\left\{-C_{nr}^{(3)} - C_{nr}^{(1)}\,\mathcal{F}_{nr}\right\}, \tag{3.15}$$

where

$$G_{nr;IJ}^{D2} = {u_I}^a{u_J}^b\eta_{ab} + 2i\bar{\theta}_+\Gamma_\mu e_{(I}{}^\mu\partial_{J)}\theta_+. \tag{3.16}$$

$\mathcal{F}_{nr}$ is the superinvariant U(1) field strength modified by the NS two form $b_{nr}^{(2)}$,

$$\mathcal{F}_{nr} = d\tilde{A} - b_{nr}^{(2)}. \tag{3.17}$$

The non-relativistic D2-brane action (3.15) obtained by dualizing x^{11} of the non-relativistic M2-brane coincides with the one obtained by taking the non-relativistic limit of the D2-brane action. The non-relativistic Dp-branes have been derived [6] by taking the non-relativistic limit of the DBI actions for general p. Here we illustrate it for the D2-brane case but starting from the Polyakov form action (3.5). To take the non-relativistic limit of the D2-brane action (3.5) we make the rescaling as in the M2-brane case, (2.6) and (2.8)

$$X^\mu = \omega x^\mu, \qquad X^a = x^a, \qquad \theta = \sqrt{\omega}\theta_- + \frac{1}{\sqrt{\omega}}\theta_+, \tag{3.18}$$

$$\gamma_{IJ} = \omega^2\tilde{\gamma}_{IJ}, \qquad T_2 = \frac{1}{\omega}\tilde{T}_2, \tag{3.19}$$

where $\theta_\pm$ is defined using the projectors for D2-brane,

$$\mathbb{P}_\pm = \frac{1}{2}(1 \pm \Gamma_*), \qquad \Gamma_* = \Gamma_0\Gamma_1\Gamma_2. \tag{3.20}$$

In addition to it we need the rescaling of the U(1) gauge field,

$$A_I = \omega \tilde{A}_I. \tag{3.21}$$

These rescalings in (3.5) lead us to

$$S^{D2} = \tilde{T}_2 \int d^3\xi \left[\omega^2 \left(\mathcal{L}^P_{\rm div} + \mathcal{L}^{WZ}_{\rm div}\right) + \left(\mathcal{L}^P_{\rm fin} + \mathcal{L}^{WZ}_{\rm fin}\right)\right], \tag{3.22}$$

where

$$\mathcal{L}^P_{\rm div} = -\frac{\sqrt{-\tilde{\gamma}}}{2}\left(\tilde{\gamma}^{IJ} g_{IJ} - 1\right), \quad \mathcal{L}^P_{\rm fin} = -\sqrt{-\tilde{\gamma}}\left(\frac{1}{2}\tilde{\gamma}^{IJ} G^{D2}_{nr;IJ} + \frac{1}{4}\tilde{\gamma}^{IK}\tilde{\gamma}^{JL}\mathcal{F}_{nr;IJ}\mathcal{F}_{nr;KL}\right), \tag{3.23}$$

where $G^{D2}_{nr;IJ}$ and $\mathcal{F}_{nr;IJ}$ ere given in (3.16) and (3.17). The divergent terms have the same form as M2-brane case, except the dimensions of the target space is 10 in the D2-brane. Here we make an alternative analysis to obtain the NG form of action. In the above Lagrangian (3.22) we regard ω sufficiently large but finite and inverse power terms of ω, which will not contribute, are omitted. Now, we remove the dependence on the worldvolume metric $\tilde{\gamma}_{IJ}$ from the Lagrangian (3.22) by solving the equation of motion of $\tilde{\gamma}_{IJ}$ as

$$\tilde{\gamma}_{IJ} = g_{IJ} + \frac{1}{\omega^2} T_{IJ}, \tag{3.24}$$

where T_{IJ} is the worldvolume energy-momentum tensor defined from $\mathcal{L}_{\rm fin}$. The ω^{-2} term in $\tilde{\gamma}_{IJ}$ could give finite contributions when the $\tilde{\gamma}_{IJ}$ in (3.24) is introduced back into the action (3.22). However, it can be shown that these terms cancel and do not contribute after all.[3] For further calculations we can take only the leading term of (3.24); $\tilde{\gamma}_{IJ} = g_{IJ}$.

Using it in the divergent term $\mathcal{L}^P_{\rm div}$ it becomes

$$d^3\xi \mathcal{L}^P_{\rm div} = -d^3\xi\sqrt{-g} = \frac{1}{3!}\varepsilon_{\mu\nu\rho} e^\mu e^\nu e^\rho. \tag{3.25}$$

Combining it with the divergent term of the WZ Lagrangian $\mathcal{L}^{WZ}_{\rm div}$ of (3.5)

$$\begin{aligned} d^3\xi \mathcal{L}_{\rm div} &= \frac{1}{3!}\varepsilon_{\mu\nu\rho} e^\mu e^\nu e^\rho - \frac{1}{2}\left\{K^-_{\mu\nu}\left[e^\mu e^\nu - K^\mu_- e^\nu + \frac{1}{3}K^\mu_- K^\nu_-\right]\right\} \\ &= \frac{1}{3!}\varepsilon_{\mu\nu\rho}\, dx^\mu dx^\nu dx^\rho. \end{aligned} \tag{3.26}$$

It is an exact form and is canceled by adding the B field Lagrangian same form as (2.9). Now, we can take the limit $\omega \to \infty$ and obtain the non-relativistic action for D2-brane,

$$\begin{aligned} S^{D2}_{nr} &= -\tilde{T}_2 \int d^3\xi \frac{\sqrt{-g}}{2}\left\{g^{IJ} G^{D2}_{nr;IJ} + \frac{1}{2} g^{IK} g^{JL}\mathcal{F}_{nr;IJ}\mathcal{F}_{nr;KL}\right\} \\ &\quad + \tilde{T}_2 \int \left\{-C^{(3)}_{nr} - C^{(1)}_{nr}\, \mathcal{F}_{nr}\right\}. \end{aligned} \tag{3.27}$$

It has the same form as (3.15) obtained by dualizing x^{11} of the non-relativistic M2-brane.

[3] It also shows that the sub-leading terms in (2.8) and (3.19), if any, do not contribute to the finite Lagrangians. See appendix C.

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4. non-relativistic M2-brane to non-relativistic IIA superstring

The reduction of the M2-brane to the IIA superstring has been discussed [7]. We first make a short review of the relativistic case. We assume the M2-brane extends in X^0, X^1 and X^{11} directions and are the longitudinal coordinates. In doing the reduction of the M2-brane to the IIA superstring we assume the X^{11} is a compact coordinate parametrized by ρ

$$X^{11}(\tau, \sigma, \rho) = \rho \tag{4.1}$$

and other supercoordinates are independent of it, $(m = 0, 1, \ldots, 9)$

$$X^m(\tau, \sigma, \rho) = X^m(\tau, \sigma), \qquad \theta(\tau, \sigma, \rho) = \theta(\tau, \sigma). \tag{4.2}$$

Using this $(j = 0, 1)$

$$\Pi^{11}_j = i\bar{\theta}\Gamma^{11}\partial_j\theta, \qquad \Pi^{11}_2 = \partial_2 X^{11} = 1 \tag{4.3}$$

and the 3×3 induced metric of the M2-brane becomes

$$G_{IJ} = \begin{pmatrix} \Pi_i{}^m\Pi_j{}^n\eta_{mn} + \Pi_i{}^{11}\Pi_{11,\mathrm{j}} & \Pi_i{}^{11} \\ \Pi_j{}^{11} & 1 \end{pmatrix} \tag{4.4}$$

It follows

$$G = \det(G_{IJ}) = \det(G^{\mathrm{IIA}}_{ij}) = G^{\mathrm{IIA}}, \tag{4.5}$$

where G^{IIA}_{ij} is 2×2 induced metric of the IIA string

$$G^{\mathrm{IIA}}_{ij} = \Pi_i{}^m\Pi_j{}^n\eta_{mn}. \tag{4.6}$$

Using them the WZ term of the M2-brane (2.5) becomes the IIA Green-Schwarz superstring action [16]

$$d\xi^3\mathcal{L}^{M2;WZ} = d\rho\, d\xi^2\, \varepsilon^{ij}(i\bar{\theta}\Gamma_m\Gamma_{11}\partial_i\theta)\left(\Pi^m_j - \frac{1}{2}(i\bar{\theta}\Gamma^m\partial_j\theta)\right) = d\xi^3\mathcal{L}^{\mathrm{IIA;WZ}}. \tag{4.7}$$

Then the M2-brane action becomes, using (4.5) and (4.7), the IIA superstring action

$$S^{\mathrm{IIA}} = T^{\mathrm{IIA}} \int d\tau d\sigma(-\sqrt{-G^{\mathrm{IIA}}} + \mathcal{L}^{\mathrm{IIA;WZ}}) \tag{4.8}$$

with the string tension

$$T^{\mathrm{IIA}} = \left(\int d\rho\right) T^{M2}. \tag{4.9}$$

The non-relativistic IIA superstring action has been obtained from the Green-Schwartz action in [4]. Here we show it is obtained by the double dimensional reduction of the non-relativistic M2-brane. The reduction is similar to the relativistic case given above but an attention should be paid for the spinor sector. The 11D fermion θ is split into the Majorana-Weyl spinors $\theta^{\pm}$ of chiralities ± 1 using $\mathbb{h}^{\pm} = \frac{1}{2}(1 \pm \Gamma_{11})$. In the non-relativistic limit each Majorana-Weyl spinors are further separated using the projection operators $\mathbb{P}_{\pm}$

in 10D. Since the longitudinal coordinates of the M2-brane are x^0, x^1 and x^{11} the projection operators $\mathbb{P}_\pm$ in (2.7) of the non-relativistic M2-brane is

$$\mathbb{P}_\pm = \frac{1}{2}(1 \pm \Gamma_*), \qquad \Gamma_* = \Gamma_0\Gamma_1\Gamma_{11}. \tag{4.10}$$

A crucial property is that the Γ_* has same form for the IIA superstring and the projection operators, $\mathbb{h}_\pm$ and $\mathbb{P}_\pm$, are commuting. It makes the non-relativistic limit and the double dimensional reduction commutative.

In doing the reduction of the non-relativistic M2-brane to the non-relativistic IIA superstring we follow the same procedure as the relativistic case. We assume x^0, x^1 and x^{11} are the longitudinal coordinates and the x^{11} is a compact coordinate parametrized by ρ

$$x^{11}(\tau, \sigma, \rho) = \rho. \tag{4.11}$$

Other supercoordinates are independent on it,

$$x^m(\tau, \sigma, \rho) = x^m(\tau, \sigma), \qquad \theta_\pm(\tau, \sigma, \rho) = \theta_\pm(\tau, \sigma). \tag{4.12}$$

Under these assumptions, for $(i = 0, 1;\ \hat{\mu} = 0, 1)$,

$$\begin{aligned}
\begin{pmatrix} e_i{}^{\hat{\mu}}, & e_2{}^{\hat{\mu}} \\ e_i{}^{11}, & e_2{}^{11} \end{pmatrix} &= \begin{pmatrix} \partial_i x^{\hat{\mu}} + i\bar{\theta}_-\Gamma^{\hat{\mu}}\partial_i\theta_-, & 0 \\ i\bar{\theta}_-\Gamma^{11}\partial_i\theta_-, & 1 \end{pmatrix}, \\
g_{IJ} &= \begin{pmatrix} g^{\rm IIA}_{ij} + e_i{}^{11}e_j{}^{11}, & e_i{}^{11} \\ e_j{}^{11} & 1 \end{pmatrix}, \qquad g^{\rm IIA}_{ij} \equiv e_i{}^{\hat{\mu}}e_j{}^{\hat{\nu}}\eta_{\hat{\mu}\hat{\nu}}, &(4.13)\\
g^{IJ} &= \begin{pmatrix} g^{ij}_{\rm IIA}, & -g^{ik}_{\rm IIA}e_k{}^{11} \\ -g^{jk}_{\rm IIA}e_k{}^{11} & 1 + g^{k\ell}_{\rm IIA}e_k{}^{11}e_\ell{}^{11} \end{pmatrix}, \qquad \det g_{IJ} = \det g^{\rm IIA}_{ij}. &(4.14)
\end{aligned}$$

where $g^{ij}_{\rm IIA}$ is the inverse of induced metric $g^{\rm IIA}_{ij}$ of the IIA superstring . For the transverse components, with the index $\hat{a} = 2, \ldots 9$,

$$\begin{aligned}
u_i{}^{\hat{a}} &= \partial_i x^{\hat{a}} + (i\bar{\theta}_-\Gamma^{\hat{a}}\partial_i\theta_+ + i\bar{\theta}_+\Gamma^{\hat{a}}\partial_i\theta_-), \qquad u_2{}^{\hat{a}} = 0, &(4.15)\\
G^{nr}_{IJ} &= \begin{pmatrix} u_i{}^{\hat{a}}u_j{}^{\hat{b}}\delta_{\hat{a}\hat{b}} + 2i\bar{\theta}_+\Gamma_{\hat{\mu}}e_{(i}{}^{\hat{\mu}}\partial_{j)}\theta_+ + 2i\bar{\theta}_+\Gamma_{11}e_{(i}{}^{11}\partial_{j)}\theta_+, & i\bar{\theta}_+\Gamma_{11}\partial_i\theta_+ \\ i\bar{\theta}_+\Gamma_{11}\partial_j\theta_+, & 0 \end{pmatrix}. &(4.16)
\end{aligned}$$

Using them in the Nambu-Goto term of the non-relativistic M2-brane Lagrangian (2.27) it becomes that of the non-relativistic IIA superstring,

$$\begin{aligned}
\mathcal{L}^{NG}_{\rm fin} &= -\frac{\sqrt{-g}}{2}\, g^{IJ}G^{nr}_{IJ} = -\frac{\sqrt{-g^{\rm IIA}}}{2}\, g^{ij}_{\rm IIA}G^{\rm IIA}_{nr;ij}, \\
G^{\rm IIA}_{nr;ij} &= u_i{}^{\hat{a}}u_j{}^{\hat{b}}\delta_{\hat{a}\hat{b}} + 2i\bar{\theta}_+\Gamma_{\hat{\mu}}e_{(i}{}^{\hat{\mu}}\partial_{j)}\theta_+. &(4.17)
\end{aligned}$$

In the WZ three form b^{M2}_{nr} of the non-relativistic M2-brane (2.28) only terms including $e_2{}^{11} = 1$ remain and

$$b^{M2}_{nr} = -d\rho\left\{K^+_{\hat{\mu}11}\left(e^{\hat{\mu}} - \frac{1}{2}K^{\hat{\mu}}_-\right) + \frac{1}{2}K^-_{\hat{\mu}11}K^{\hat{\mu}}_+ + L_{\hat{b}11}\left(u^{\hat{b}} - \frac{1}{2}L^{\hat{b}}\right)\right\}. \tag{4.18}$$

Using a relation $\Gamma_{\hat{\mu}}\Gamma_{11}\theta_\pm = \pm\varepsilon_{\hat{\mu}\hat{\nu}}\Gamma^{\hat{\nu}}\theta_\pm$ in (A.11) and the IIA cyclic identity (A.5) it becomes the WZ Lagrangian of the non-relativistic IIA superstring given in [4] up to an exact form. Then the non-relativistic M2-brane action is reduced to that of the non-relativistic IIA superstring

$$S_{nr}^{\rm IIA} = \tilde{T}^{\rm IIA}\int d\tau d\sigma \Big\{ -\frac{e}{2}\, g_{\rm IIA}^{jk}\; u_j{}^{\hat{a}} u_k{}^{\hat{b}}\delta_{\hat{a}\hat{b}} \;-\; 2i\, e\,(\overline{\theta}_+\Gamma^{\hat{\mu}} e_{\hat{\mu}}{}^i\partial_i\theta_+) \\ -2i\,\varepsilon^{jk}\left(\overline{\theta}_+\Gamma_{\hat{a}}\Gamma_{11}\,\partial_j\theta_-\right)\left(u_k{}^{\hat{a}} - i\,\overline{\theta}_+\Gamma^{\hat{a}}\,\partial_k\theta_-\right)\Big\}\,, \tag{4.19}$$

where e and $e_{\hat{\mu}}{}^i$ are the determinant and inverse of $e_i{}^{\hat{\mu}}$ respectively and

$$\tilde{T}^{\rm IIA} = \left(\int d\rho\right)\tilde{T}^{M2}. \tag{4.20}$$

5. T-duality of non-relativistic D2 and D1 branes

It is known Dp-branes are T-dual of D$(p+1)$-branes [9, 10]. The T-duality covariance of the worldvolume actions of D-branes is studied in detail [17, 18]. The T-dualities of the non-relativistic branes have been discussed in [1]. In this section we discuss the T-duality as that of the non-relativistic D-brane worldvolume actions. Although our results can be applied to general Dp-branes we confine our discussions to the case of the T-duality between D2 and D1 branes for definiteness. There is a subtlety in the non-relativistic limit since the D2-brane belongs to IIA sector while D1-string does to IIB sector and there is a chirality flip of a Majorana-Weyl fermion. In contrast to the case of reduction from the M2 to the IIA superstring discussed in the last section, the chirality projectors $\mathbb{h}^\pm$ and non-relativistic projectors $\mathbb{P}_\pm$ do not commute. Nevertheless we can show that the T-duality transformations and non-relativistic limits are commutative and the non-relativistic D1-string can be obtained from non-relativistic D2-brane by the T-duality.

D2 to D1. In the relativistic D2-brane X^0, X^1 and X^2 are the longitudinal coordinates and depend on $\xi^I = (\tau,\sigma,\rho)$. In doing reduction to D1 we assume one of the longitudinal coordinate, X^2, is compact and parametrized by ρ

$$X^2(\tau,\sigma,\rho) = \rho \tag{5.1}$$

while others $X^m, (m \neq 2), A_I$ and θ are functions of τ and σ only. The dynamical degree of freedom, X'^2, of D1 coordinate is supplied from that of a U(1) gauge field component A_2,[4]

$$X'^2 = A_2, \tag{5.2}$$

and others are

$$X'^m = X^m, \quad (m \neq 2) \qquad A'_j = A_j, \quad (j = 0,1). \tag{5.3}$$

[4]In this section primed variables are those of D1 and unprimed ones are of D2. The variables in the non-relativistic limit are indicated with tildes.

The D2(IIA) spinors are related to those of D1(IIB) by flipping one of chirality component

$$\theta' = \begin{pmatrix} \theta'^1 \\ \theta'^2 \end{pmatrix} = \left(\frac{1+\tau_3}{2} + \frac{1-\tau_3}{2}\Gamma_2 \right) \begin{pmatrix} \theta^+ \\ \theta^- \end{pmatrix} = \begin{pmatrix} \mathbb{h}^+\theta \\ \Gamma_2\, \mathbb{h}^-\theta \end{pmatrix}. \tag{5.4}$$

Both IIB spinors θ'^1 and θ'^2 are Majorana-Weyl fermions with same chirality $+$. Under this T-duality transformation the worldvolume action of the D2-brane (3.8) turns out to be that of the D1-string,

$$S^{D1} = -T_1 \int d^2\xi \sqrt{-\det(G^{D1}_{ij} + \mathcal{F}'_{ij})} - T_1 \int C^{(2)}, \tag{5.5}$$

where

$$G^{D1}_{ij} = \Pi'^m_i \Pi'^n_j \eta_{mn}, \qquad \Pi'^m = dX'^m + i\overline{\theta}'\Gamma^m\theta', \tag{5.6}$$

$$\mathcal{F}' = dA' - i\overline{\theta}'\Gamma^m\tau_3\theta' \left(dX'^m + \frac{i}{2}\overline{\theta}'\Gamma^m\theta' \right), \quad C^{(2)} = i\overline{\theta}'\Gamma^m\tau_1\theta' \left(dX'^m + \frac{i}{2}\overline{\theta}'\Gamma^m\theta' \right). \tag{5.7}$$

Although the transformation breaks manifest Lorentz covariance the resulting action is superPoincare invariant and kappa invariant [17, 18].

D1 to non-relativistic D1. In the non-relativistic limit the D1-string variables are rescaled as

$$\widetilde{(x'^{\hat\mu})} = \frac{1}{\omega} X'^{\hat\mu}, \quad (\hat\mu = 0,1), \qquad \widetilde{(A'_j)} = \frac{1}{\omega} A'_j, \quad (j = 0,1), \tag{5.8}$$

$$\widetilde{(\theta')} = \left(\sqrt{\omega}\, \mathbb{P}'_+ + \frac{1}{\sqrt{\omega}}\, \mathbb{P}'_- \right) \theta', \qquad \tilde{T}_1 = T_1, \tag{5.9}$$

where $\mathbb{P}'_\pm$ are the non-relativistic projectors for D1-string,

$$\mathbb{P}'_\pm = \frac{1}{2}(1 \pm \Gamma'_*), \qquad \Gamma'_* = \Gamma_{01}\tau_1. \tag{5.10}$$

By taking $\omega \to \infty$ limit we obtain non-relativistic D1-string action [6]. Abbreviating tilde and prime indices for the non-relativistic D1 variables,

$$\begin{aligned} S^{D1}_{nr} = \tilde{T}^{D1} \int d\tau d\sigma \Big\{ &-\frac{e}{2}\, g^{jk}\, u_j{}^{\hat a} u_k{}^{\hat b} \delta_{\hat a \hat b} \;-\; 2i\, e\, (\overline{\theta}_+ \Gamma^{\hat\mu} e_{\hat\mu}{}^i \partial_i \theta_+) - \frac{e}{4} g^{ij} g^{k\ell} \mathcal{F}^{D1}_{nr;ik} \mathcal{F}^{D1}_{nr;j\ell} \\ &-2i\, \varepsilon^{jk} \left(\overline{\theta}_+ \Gamma_{\hat a}\tau_1\, \partial_j \theta_- \right) \left(u_k{}^{\hat a} - i\, \overline{\theta}_+ \Gamma^{\hat a}\, \partial_k \theta_- \right) \Big\}. \end{aligned} \tag{5.11}$$

Comparing to the non-relativistic IIA superstring (4.19) Γ_{11} is replaced by τ_1 in the WZ term and $\mathcal{F}^{D1}_{nr}$ is $\mathcal{F}_{nr}$ of (3.17) in which Γ_{11} is replaced by τ_3.

D2 to non-relativistic D2. On the other hand starting from the relativistic D2-brane action (3.8) we have obtained the non-relativistic D2-brane action (3.27) in the non-relativistic limit. The non-relativistic rescalings are (3.19) and (3.21),

$$\tilde{x}^\mu = \frac{1}{\omega}\, X^\mu\,, \quad (\mu = 0,1,2), \qquad \tilde{A}_I = \frac{1}{\omega}\, A_I, \quad (I = 0,1,2) \tag{5.12}$$

and

$$\tilde\theta = \left(\sqrt{\omega}\, \mathbb{P}_+ + \frac{1}{\sqrt{\omega}}\, \mathbb{P}_- \right) \theta, \qquad \tilde{T}_2 = \omega\, T_2, \tag{5.13}$$

where $\mathbb{P}_\pm$ are the projectors for D2-brane,

$$\mathbb{P}_\pm = \frac{1}{2}(1 \pm \Gamma_*), \qquad \Gamma_* = \Gamma_{012}. \tag{5.14}$$

Non-relativistic D2 to non-relativistic D1. From the non-relativistic D2-brane to non-relativistic D1-string we make the same T-duality transformation as the relativistic case. Compactify in the longitudinal direction $\tilde{x}^2$ and choose

$$\tilde{x}^2(\tau,\sigma,\rho)=\rho \tag{5.15}$$

while others are independent of ρ. The dynamical degree of freedom, $(\tilde{x}^2)'$, of non-relativistic D1 string coordinate is transferred from that of a U(1) gauge field component $\tilde{A}_2$,

$$(\tilde{x}^2)'{=}\tilde{A}_2, \tag{5.16}$$

and others are

$$(\tilde{x}^m)'{=}\tilde{x}^m, \quad (m\neq 2), \qquad (\tilde{A}_j)'{=}\tilde{A}_j, \quad (j=0,1). \tag{5.17}$$

The non-relativistic D2(IIA) spinors are related to those of non-relativistic D1(IIB) by flipping one of chirality component

$$(\tilde{\theta})'=\begin{pmatrix} \mathbb{h}^+\tilde{\theta} \\ \Gamma_2\,\mathbb{h}^-\tilde{\theta}\end{pmatrix}=\begin{pmatrix} \mathbb{h}^+(\sqrt{\omega}\,\mathbf{P}_+ + \frac{1}{\sqrt{\omega}}\,\mathbf{P}_-)\theta \\ \Gamma_2\,\mathbb{h}^-(\sqrt{\omega}\,\mathbf{P}_+ + \frac{1}{\sqrt{\omega}}\,\mathbf{P}_-)\theta\end{pmatrix}. \tag{5.18}$$

They show that the T-duality transformations and the non-relativistic limit are commutative especially

$$(\tilde{\theta})'{=}\widetilde{(\theta')} \qquad \text{and} \qquad (\tilde{x}^2)'{=}\widetilde{(x^{2\prime})}. \tag{5.19}$$

Note the latter comes from the fact that both the longitudinal coordinates X^μ and the U(1) gauge field A_I are rescaled in a same power of ω under the non-relativistic rescaling. The commutativity guarantees that the non-relativistic D1-string action obtained by the T-duality transformation of the non-relativistic D2-brane coincides with that given by the non-relativistic limit of the D1-string. Actually making the transformations on the non-relativistic D2 brane action (3.27) we can obtain non-relativistic D1 string action (5.11).

6. Summary

In this paper we have analyzed, at the world volume level, several corners of M-theory described by non-relativistic theories. In particular we have studied how these non-relativistic theories are mapped to each other by the duality symmetries of the worldvolume actions. In particular we have analyzed how the non-relativistic M2-brane is related via these dualities to non-relativistic D2-brane, non-relativistic IIA fundamental string and also by using T-duality to non-relativistic D1-string. Their worldvolume actions coincide with those obtained from the relativistic actions by taking the non-relativistic limit [4, 6]. Thus we have shown that the non-relativistic limit and the duality transformations are commutative as illustrated in the figure 1.

From a more technical point of view we have constructed non-relativistic branes starting from the Polyakov formulation of relativistic M2 and D2-brane. One can also show that the kappa symmetry is maintained in the non-relativistic limit and it is also preserved through the use of duality symmetries of M theory. Since non-relativistic D-branes in the

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static gauge and $\theta_- = 0$ are described by free supersymmetric gauge theories it would be interesting to see how these M dualities are realized on these supersymmetric gauge theories.

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A. Notations and some useful formulae

Here we summarize some notations. Indices are

$$\begin{array}{lcc} & \underline{10\ \text{dim}} & \underline{11\ \text{dim}} \\ \text{target space}: & m,n=0,\ldots,9 & M,N=0,\ldots,10 \end{array}$$

$$\begin{array}{rcc} & \underline{p=1} & \underline{p=2} \\ \text{target space, longitudinal}: & \hat\mu,\hat\nu=0,1 & \mu,\nu=0,1,2 \\ \text{target space, transverse}: & \hat a,\hat b=2,\ldots,9 & a,b=3,\ldots,9 \\ \text{worldvolume}: & i,j=0,1 & I,J=0,1,2 \end{array}$$

and we use $A,B=3,\ldots,9,11$ for transverse indices when we work in 11 dimensions. The metrics of target space and worldvolume have mostly + signatures. The totally antisymmetric Levi-Civita tensor is normalized by $\varepsilon^{012\ldots p}=+1$, and $\varepsilon_{012\ldots p}=-1$.

The gamma matrices are Γ^m and $\Gamma_{11}=\Gamma_0\Gamma_1\ldots\Gamma_9$. They can be chosen real by taking the charge conjugation matrix $C=\Gamma_0$ and satisfy

$$C^{-1}\Gamma^M C=-\,(\Gamma^M)^T, \qquad C^T=-C. \tag{A.1}$$

The conjugate $\overline{\theta}$ of the Majorana spinor is defined by $\overline{\theta}=\theta^T C$. For type IIA theories θ is a Majorana spinor while for type IIB theories there are two Majorana-Weyl spinors θ_α $(\alpha=1,2)$ of the same chirality. The index α on which the Pauli matrices τ_1,τ_2,τ_3 act is not displayed explicitly. This leads to some useful symmetry relations as

$$\bar\chi\lambda=\bar\lambda\chi, \qquad \lambda=\Gamma_m\epsilon \ \rightarrow\ \bar\lambda=-\bar\epsilon\Gamma_m, \qquad \lambda=\Gamma_{11}\epsilon \ \rightarrow\ \bar\lambda=-\bar\epsilon\Gamma_{11}. \tag{A.2}$$

In 11-dimensions we have the gamma matrix identity

$$(C\Gamma_{ML})_{(\alpha\beta}(C\Gamma^L)_{\gamma\delta)}=0, \tag{A.3}$$

where all indices $\alpha,\beta,\gamma,\delta$ are symmetrized. Therefore

$$(d\overline{\theta}\Gamma_{ML}d\theta)(\overline{\theta}\Gamma^L d\theta)=-(\overline{\theta}\Gamma_{ML}d\theta)(d\overline{\theta}\Gamma^L d\theta). \tag{A.4}$$

In 10-dimensions the gamma matrix identity is

$$(C\Gamma_m\Gamma_{11})_{\alpha(\beta}(C\Gamma^m)_{\gamma\delta)} + (C\Gamma_m)_{\alpha(\beta}(C\Gamma^m\Gamma_{11})_{\gamma\delta)} = 0, \tag{A.5}$$

where Γ_{11} can be replaced by τ_i $(i = 1, 3)$ for type IIB spinors.

From this gamma matrix identity we can derive cyclic identities in 10-dimensions

$$\sum_{IJK\ \mathrm{cyclic}} \left[\Gamma_m\theta_I\ (\bar\theta_J\Gamma^m\ \theta_K) + \Gamma_m\Gamma_{11}\theta_I\ (\bar\theta_J\Gamma^m\Gamma_{11}\ \theta_K)\right] = 0, \tag{A.6}$$

for the type IIA spinors and

$$\sum_{IJK\ \mathrm{cyclic}} \left\{\Gamma_m\tau_i\theta_I\left(\bar\theta_J\Gamma^m\theta_K\right) + \Gamma_m\theta_I\left(\bar\theta_J\Gamma^m\tau_i\theta_K\right)\right\} = 0, \qquad (i = 1, 3). \tag{A.7}$$

for type IIB spinors.

In the non-relativistic brane theories spinors are separated by projection operators

$$\mathbb{P}_\pm = \frac{1}{2}\left(1 \pm \Gamma_*\right), \qquad \theta_\pm = \mathbb{P}_\pm\theta \tag{A.8}$$

where Γ_* is defined for each systems as

$$\begin{aligned} \text{IIA string}: \ \ & \Gamma_* = \Gamma_0\Gamma_1\Gamma_{11} \\ \text{IIB string}: \ \ & \Gamma_* = \Gamma_0\Gamma_1\tau_3 \\ \text{D1}: \ \ & \Gamma_* = \Gamma_0\Gamma_1\tau_1 \\ \text{D2}: \ \ & \Gamma_* = \Gamma_0\Gamma_1\Gamma_2 \\ \text{M2}: \ \ & \Gamma_* = \Gamma_0\Gamma_1\Gamma_{11}. \end{aligned} \tag{A.9}$$

For the D2-brane there is a relation

$$\Gamma_{\mu\nu}\theta_\pm = \mp\varepsilon_{\mu\nu\rho}\Gamma^\rho\theta_\pm \tag{A.10}$$

and

$$\Gamma_{\hat\mu}\Gamma_{11}\theta_\pm = \pm\varepsilon_{\hat\mu\hat\nu}\Gamma^{\hat\nu}\theta_\pm, \qquad \Gamma_{\hat\mu}\tau_1\theta_\pm = \pm\varepsilon_{\hat\mu\hat\nu}\Gamma^{\hat\nu}\theta_\pm \tag{A.11}$$

for the F1 and D1 strings.

The differential forms and the spinors have independent gradings. For $A^{p,r}$,$B^{q,s}$ of the form grades p, q and the Grassman parities r, s

$$A^{p,r}B^{q,s} = (-1)^{pq+rs}\ B^{q,s}A^{p,r}. \tag{A.12}$$

Components of the forms are defined by

$$A_r = \frac{1}{r!}A_{i_1\ldots i_r}\mathrm{d}\xi^{i_1}\ldots\mathrm{d}\xi^{i_r}, \tag{A.13}$$

and differentials are taken from the left.

B. DBI action from D2-brane Polyakov action

We have seen that the action of D2-brane in the Polyakov form is obtained by dualizing X^{11} of the M2-brane action, (3.5). The kinetic term is

$$S^{\rm Pol} = -T_2 \int d^3\xi \frac{\sqrt{-\gamma}}{2} \left\{ \gamma^{IJ} G^{D2}_{IJ} - 1 + \frac{1}{2} \gamma^{IK} \gamma^{JL} \mathcal{F}_{IJ} \mathcal{F}_{KL} \right\} \tag{B.1}$$

while the WZ term does not depends on the metric γ_{IJ}. Here we give a proof that it gives the DBI action (3.8)

$$S^{BI} = -T_2 \int d^3\xi \sqrt{-\det(G^{D2}_{IJ} + \mathcal{F}_{IJ})} \tag{B.2}$$

by integrating the worldvolume metric. In order to show it we should solve the equation of motion of the worldvolume metric and put it back to the action to eliminate its dependence.

The equation of motion by taking variation of the action (B.1) with respect to γ_{IJ}, is

$$\gamma_{IJ} = \frac{2}{\mathcal{X} - \mathcal{Y} - 1} \left(G^{D2}_{IJ} - \mathcal{F}_{IK} \gamma^{KL} \mathcal{F}_{LJ} \right), \tag{B.3}$$

where we have defined

$$\mathcal{X} = \gamma^{IJ} G^{D2}_{IJ}, \qquad \mathcal{Y} = \frac{1}{2} \gamma^{IJ} \mathcal{F}_{JK} \gamma^{KL} \mathcal{F}_{LI}. \tag{B.4}$$

Since we are in 3 worldvolume dimensions it is convenient to introduce independent three vector $\mathcal{B}^K$ by

$$\mathcal{F}_{IJ} = \varepsilon_{IJK} \mathcal{B}^K \tag{B.5}$$

so that (B.3) is

$$\gamma_{IJ} = \frac{2}{\mathcal{X} - \mathcal{Y} - 1} \left(G^{D2}_{IJ} + \frac{1}{\det\gamma} \{ \gamma_{IJ} (\mathcal{B}^K \gamma_{KL} \mathcal{B}^L) - (\gamma_{IK} \mathcal{B}^K)(\gamma_{JL} \mathcal{B}^L) \} \right), \tag{B.6}$$

and $\mathcal{Y}$ in (B.4) is

$$\mathcal{Y} = -\frac{(\mathcal{B}^I \gamma_{IJ} \mathcal{B}^J)}{\det\gamma} \equiv -\frac{(\mathcal{B}\gamma\mathcal{B})}{\gamma} \tag{B.7}$$

Multiplying γ^{IJ} on (B.6)

$$3 = \frac{2}{\mathcal{X} - \mathcal{Y} - 1} \left(\mathcal{X} + \frac{2}{\gamma} (\mathcal{B}\gamma\mathcal{B}) \right). \tag{B.8}$$

Multiplying $\mathcal{B}^I \mathcal{B}^J$ on (B.6)

$$(\mathcal{B}\gamma\mathcal{B}) = \frac{2}{\mathcal{X} - \mathcal{Y} - 1} \left(\mathcal{B} G^{D2} \mathcal{B} \right). \tag{B.9}$$

Solving (B.7) and (B.8) for $\mathcal{X}$ and $(\mathcal{B}\gamma\mathcal{B})$ as

$$\mathcal{X} = 3 - \mathcal{Y}, \qquad (\mathcal{B}\gamma\mathcal{B}) = -\gamma\, \mathcal{Y} \tag{B.10}$$

and using them in (B.9)

$$\gamma\,\mathcal{Y}(1-\mathcal{Y})+(\mathcal{B}G^{D2}\mathcal{B})=0. \tag{B.11}$$

Return them back to (B.6)

$$\gamma_{IJ}=G^{D2}_{IJ}-\frac{1}{\gamma\,(1-\mathcal{Y})^2}(G^{D2}_{IK}\mathcal{B}^K)(G^{D2}_{JL}\mathcal{B}^L), \tag{B.12}$$

and computing $\gamma=(\det\gamma)$ we get

$$\gamma=G^{D2}-\frac{G^{D2}(\mathcal{B}G^{D2}\mathcal{B})}{\gamma\,(1-\mathcal{Y})^2}=0. \tag{B.13}$$

(B.11) and (B.13) are solved as

$$\mathcal{Y}=-\frac{(\mathcal{B}G^{D2}\mathcal{B})}{G^{D2}},\qquad \gamma=\frac{(G^{D2})^2}{(\mathcal{B}G^{D2}\mathcal{B})+G^{D2}} \tag{B.14}$$

and we determine γ_{IJ} in terms of G^{D2}_{IJ} and $\mathcal{B}^I$ as

$$\gamma_{IJ}=G^{D2}_{IJ}-\frac{(G^{D2}_{IK}\mathcal{B}^K)(G^{D2}_{JL}\mathcal{B}^L)}{(\mathcal{B}G^{D2}\mathcal{B})+G^{D2}}. \tag{B.15}$$

Finally we use it in the Lagrangian density of (B.1). By squaring it becomes

$$\mathcal{L}^2=\frac{-\gamma}{4}(\mathcal{X}-1-\mathcal{Y})=-\left(G^{D2}+(\mathcal{B}G^{D2}\mathcal{B})\right)=-\det(G^{D2}_{IJ}+\mathcal{F}_{IJ}). \tag{B.16}$$

and the DBI action (B.2) follows from it.

C. ω^{-2} order term of γ_{IJ} does not contribute

As we have seen in section 3 divergent term of $\mathcal{L}^P$ (3.23) is

$$\mathcal{L}^P_{\rm div}=-\frac{\sqrt{-\tilde{\gamma}}}{2}\left(\tilde{\gamma}^{IJ}g_{IJ}-1\right), \tag{C.1}$$

and on the other hand the $\tilde{\gamma}_{ij}$ from the Lagrange equation of (3.22) is

$$\tilde{\gamma}_{IJ}=g_{IJ}+\frac{1}{\omega^2}T_{IJ}+\mathcal{O}\left(\omega^{-4}\right) \tag{C.2}$$

A priori, we can expect that the ω^{-2} order term of this expansion will give finite contribution in the divergent $\omega^2\mathcal{L}^P_{\rm div}$. However it is easy to show that this is not what happens. Introducing the expansion of $\tilde{\gamma}_{IJ}$ in $\mathcal{L}^P_{\rm div}$

$$\begin{aligned}\mathcal{L}^P_{\rm div}&=-\frac{\sqrt{-g}}{2}\left(1+\frac{1}{\omega^2}g^{IJ}T_{IJ}\right)^{\frac{1}{2}}\left((g^{IJ}-\frac{1}{\omega^2}T^{IJ})g_{IJ}-1\right)\\&=-\frac{\sqrt{-g}}{2}\left(1+\frac{1}{2\omega^2}g^{IJ}T_{IJ}\right)\left(2-\frac{1}{\omega^2}T^{IJ}g_{IJ}\right)\\&=-\sqrt{-g}+\mathcal{O}\left(\omega^{-4}\right)\end{aligned} \tag{C.3}$$

where $T^{IJ}=g^{IK}T_{KL}g^{LJ}$. It shows that ω^{-2} order term of $\tilde{\gamma}_{IJ}$ in (C.2) does not give finite contribution in the divergent Lagrangian $\omega^2\mathcal{L}^P_{\rm div}$.

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